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## ON IMMERSED ORIENTED SURFACES AND THEIR PLANE PROJECTIONS

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### 1. INTRODUCTION

Throughout the report, all manifolds and maps are differentiable of class  $C^\infty$ . Let  $M$  be a closed connected surface,  $f$  and  $g : M \rightarrow \mathbb{R}^3$  immersions. Let  $F : M \times [0, 1] \rightarrow \mathbb{R}^3$  be a homotopy between  $f$  and  $g$ . That is,  $F|_{M \times \{0\}} = f$  and  $F|_{M \times \{1\}} = g$  hold. We call  $F$  a regular homotopy between  $f$  and  $g$  if a level preserving map  $(F \times \text{id}) : M \times [0, 1] \rightarrow \mathbb{R}^3 \times [0, 1]$  defined by  $(F \times \text{id})(x, t) = (F(x, t), t)$  is an immersion. James and Thomas [4] proved that the space of immersions  $f : M \rightarrow \mathbb{R}^3$  has  $2^{\dim H_1(M; \mathbb{Z}_2)}$  connected components. Pinkall [6] associated to any immersion  $f : M \rightarrow \mathbb{R}^3$  a  $\mathbb{Z}_4$ -valued quadratic form  $q_f : H_1(M; \mathbb{Z}_2) \rightarrow \mathbb{Z}_4$  and proved that two immersions  $f$  and  $g : M \rightarrow \mathbb{R}^3$  are regularly homotopic if and only if  $q_f = q_g$  holds.

Let  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be an orthogonal projection and  $\tilde{f} : M \rightarrow \mathbb{R}^2$  a stable map. If a stable map  $\tilde{f} : M \rightarrow \mathbb{R}^2$  has an immersion  $f : M \rightarrow \mathbb{R}^3$  such that  $\pi \circ f = \tilde{f}$ , we call that  $\tilde{f}$  has an immersion lift  $f$  and that  $f$  is an immersion lift over  $\tilde{f}$ . Let  $f$  and  $g : M \rightarrow \mathbb{R}^3$  be immersion lifts over  $\tilde{f} : M \rightarrow \mathbb{R}^2$ . If there exists a regular homotopy  $F : M \times [0, 1] \rightarrow \mathbb{R}^3$  between  $f$  and  $g$  which satisfies that  $\pi \circ (F|_{M \times \{t\}}) = \tilde{f}$  for any  $t \in [0, 1]$ , then we call that  $f$  and  $g$  are  $\tilde{f}$ -regularly homotopic.

In this report, when  $M$  is a closed connected oriented surface, we study  $\tilde{f}$ -regular homotopy classes for a fixed stable map  $\tilde{f} : M \rightarrow \mathbb{R}^2$  which has an immersion lift.

This report is organized as follows. In Section 2, we give the definition and see properties of a stable map  $\tilde{f} : M \rightarrow \mathbb{R}^2$ . In Section 3, we restate Haefliger's theorem [3] by putting sings on the apparent contour of  $\tilde{f}$ . In Section 4, we determine  $\tilde{f}$ -regular homotopy classes for a fixed stable map  $\tilde{f}$  which has an immersion lift. In Section 5, we give the definitions of generic homotopy and regular homotopy lift. By using the method obtained in the previous section, we introduce a necessary and sufficient condition for existence of regular homotopy lift over the given generic homotopy. As an application, we construct a generic homotopy whose regular homotopy lift corresponds to a sphere eversion.

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## 2. STABLE MAP

In this section, we give the definition and see properties of a stable map.

Let  $\tilde{f} : M \rightarrow \mathbb{R}^2$  be a smooth map of a closed connected surface  $M$  into the plane. We denote the set of such maps by  $C^\infty(M, \mathbb{R}^2)$ , which is equipped with the Whitney  $C^\infty$ -topology. A smooth map  $\tilde{f}$  is said to be a stable map if in  $C^\infty(M, \mathbb{R}^2)$ , there exists an open neighborhood  $U$  of  $\tilde{f}$  such that for any  $\tilde{g} \in U$ ,  $\tilde{g}$  is  $C^\infty$  right-left equivalent to  $\tilde{f}$ , i.e., there exist two diffeomorphisms  $\Phi : M \rightarrow M$  and  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & M \\ \tilde{f} \downarrow & & \downarrow \tilde{g} \\ \mathbb{R}^2 & \xrightarrow{\varphi} & \mathbb{R}^2 \end{array}$$

is commutative.

For a smooth map  $\tilde{f} : M \rightarrow \mathbb{R}^2$ , we denote by  $S(\tilde{f})$  the set of the points in  $M$  where the rank of the differential of  $\tilde{f}$  is strictly less than two. We say that  $S(\tilde{f}) \subset M$  is a singular set of  $\tilde{f}$  and  $\tilde{f}(S(\tilde{f})) \subset \mathbb{R}^2$  is an apparent contour of  $\tilde{f}$ .

The following characterizations of stable maps are well-known (see [2, 8], for example).

**Proposition 2.1.** *A smooth map  $\tilde{f} : M \rightarrow \mathbb{R}^2$  of a closed surface  $M$  is a stable map if and only if the following conditions are satisfied.*

- (i) *For every  $q \in M$ , there exist local coordinates  $(x, y)$  and  $(X, Y)$  around  $q \in M$  and  $\tilde{f}(q) \in \mathbb{R}^2$  respectively such that one of the following holds:*
  - (a)  $(X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y)$  ( $q$  : regular point),
  - (b)  $(X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y^2)$  ( $q$  : fold point),
  - (c)  $(X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y^3 - xy)$  ( $q$  : cusp point).
- (ii) *If  $q \in M$  is a cusp point, then  $\tilde{f}^{-1}(\tilde{f}(q)) \cap S(\tilde{f}) = \{q\}$ .*
- (iii) *The map  $\tilde{f}|(S(\tilde{f}) \setminus \{\text{cusp points}\})$  is an immersion with normal crossings.*

Note that  $S(\tilde{f})$  is a compact 1-dimensional submanifold of  $M$  and the number of cusp points is finite. Let  $U \subset M$  be a tubular neighborhood of  $S(\tilde{f})$ . Then the restriction of  $\tilde{f}$  on the closure  $\text{cl}(M \setminus U)$  is an immersion. By the apparent contour of  $\tilde{f}$ ,  $\mathbb{R}^2$  is naturally stratified into 2-, 1- and 0-dimensional strata. The union of 1- and 0-dimensional strata forms  $\tilde{f}(S(\tilde{f}))$ . On each 1-dimensional stratum, we can define an orientation as follows. We fix the canonical orientation on  $\mathbb{R}^2$ . Let  $\Omega$  be a connected component of  $\mathbb{R}^2 \setminus \tilde{f}(S(\tilde{f}))$ . We associate to  $\Omega$  a non-negative integer  $n_{\tilde{f}}(\Omega)$ , which is the number of points in the fiber of  $\tilde{f}$  over any point of  $\Omega$ . Every 1-dimensional stratum is adjacent to exactly two connected components of  $\mathbb{R}^2 \setminus \tilde{f}(S(\tilde{f}))$ . Since these two components have distinct  $n_{\tilde{f}}(\Omega)$ -values, we can orient each 1-dimensional stratum in  $\tilde{f}(S(\tilde{f}))$  so that the region with the larger  $n_{\tilde{f}}(\Omega)$ -value is on its left. Since  $\tilde{f}|(S(\tilde{f}) \setminus \{\text{cusp points}\})$  is an immersion,  $S(\tilde{f}) \setminus \{\text{cusp points}\}$  is also oriented.

Suppose that  $M$  is an oriented closed surface and  $\mathbb{R}^2$  is oriented plane. Let  $q$  be a cusp point of a stable map  $\tilde{f} : M \rightarrow \mathbb{R}^2$ . For a sufficiently small neighborhood  $U$  of  $f(q)$ , the map  $\tilde{f}|_V : V \rightarrow U$  has degree  $\pm 1$ , where  $V$  is the component of  $\tilde{f}^{-1}(U)$  containing  $q$ . We call  $q$  is a positive (resp. negative) cusp if the local degree of  $\tilde{f}$  at  $q$  equals  $+1$  (resp.  $-1$ ).

### 3. IMMERSION LIFT

In the following, we assume that  $M$ ,  $\mathbb{R}^3$  and  $\mathbb{R}^2$  are oriented. In this case, Haefliger's theorem is restated as follows.

**Theorem 3.1** (Haefliger [3]). *A stable map  $\tilde{f} : M \rightarrow \mathbb{R}^2$  has an immersion lift if and only if each connected component of  $S(\tilde{f})$  has even number of cusp points.*

Let  $\tilde{f} : M \rightarrow \mathbb{R}^2$  be a stable map which has an immersion lift. On each connected component of fold points  $S(\tilde{f}) \setminus \{\text{cusp points}\}$ , we can put a sign  $+1$  or  $-1$  which satisfies the following rule.

- Let  $C$  and  $C'$  be two connected components which adjacent to the same cusp point. Then  $C$  and  $C'$  have the opposite signs.

If a sign of  $C$  is  $+1$  (resp.  $-1$ ), we call  $C$  a positive (resp. negative) fold and a sign can be put on each image of fold component. Such a stable map  $\tilde{f}$  is called a signed stable map.

Let  $\tilde{f} : M \rightarrow \mathbb{R}^2$  be a signed stable map and  $U$  a tubular neighborhood of  $S(\tilde{f})$ . Since  $M$  is oriented,  $U \setminus S(\tilde{f})$  is divided into two regions  $U_+$  and  $U_-$  where  $\tilde{f}|_{U_+}$  (resp.  $\tilde{f}|_{U_-}$ ) is an orientation preserving (resp. reversing) immersion. We construct an immersion lift  $f_U : U \rightarrow \mathbb{R}^3$  over  $\tilde{f}|_U$  which satisfies the following.

- (1) If  $C$  is a positive fold,  $f$  is defined as Figure 1(a).
- (2) If  $C$  is a negative fold,  $f$  is defined as Figure 1(b).
- (3) If  $q$  is a positive cusp and negative fold comes in  $q$  for the orientation of  $S(\tilde{f})$ ,  $f$  is defined as Figure 2(a).
- (4) If  $q$  is a positive cusp and positive fold comes in  $q$  for the orientation of  $S(\tilde{f})$ ,  $f$  is defined as Figure 2(b).
- (5) If  $q$  is a negative cusp and negative fold comes in  $q$  for the orientation of  $S(\tilde{f})$ ,  $f$  is defined as Figure 2(c).
- (6) If  $q$  is a negative cusp and positive fold comes in  $q$  for the orientation of  $S(\tilde{f})$ ,  $f$  is defined as Figure 2(d).

**Definition 3.2.** Let  $\tilde{f} : M \rightarrow \mathbb{R}^2$  be a signed stable map and  $U$  a tubular neighborhood of  $S(\tilde{f})$ . If an immersion  $f : M \rightarrow \mathbb{R}^3$  satisfies the above rules (1)–(6) on  $f|_U$ , we call  $f$  an immersion lift over the signed stable map  $\tilde{f}$ .

### 4. $\tilde{f}$ -REGULAR HOMOTOPY

In this section, we state that  $\tilde{f}$ -regular homotopy classes can be determined.

**Theorem 4.1.** *If  $f$  and  $g : M \rightarrow \mathbb{R}^3$  are immersion lifts over the signed stable map  $\tilde{f} : M \rightarrow \mathbb{R}^2$ , then  $f$  and  $g$  are  $\tilde{f}$ -regularly homotopic.*

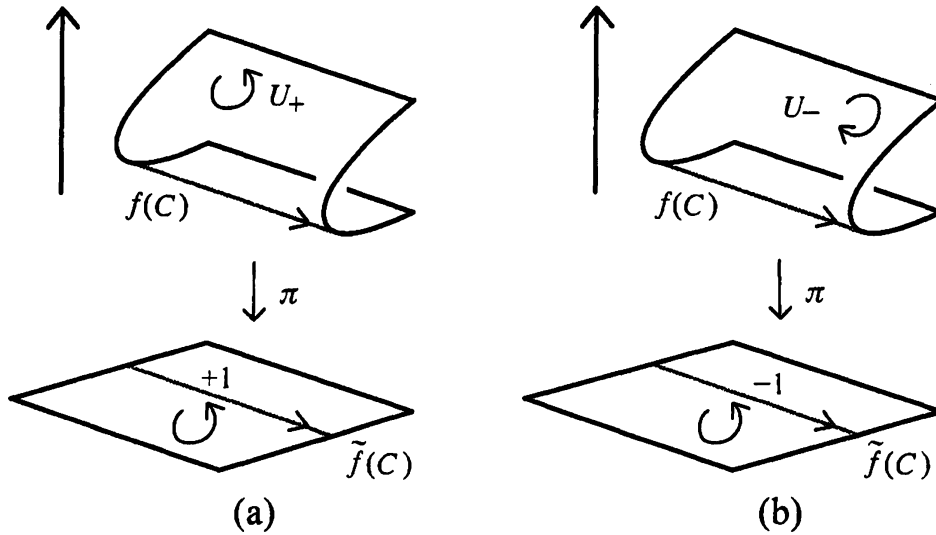


FIGURE 1. Immersion lifts if (a)  $C$  is a positive fold, (b)  $C$  is a negative fold.

**Corollary 4.2.** *Let  $\tilde{f} : M \rightarrow \mathbb{R}^2$  be a stable map which has an immersion lift. (Note that  $\tilde{f}$  is not signed.) The number of  $\tilde{f}$ -regular homotopy classes is  $2^{\sharp S(\tilde{f})}$ , where  $\sharp S(\tilde{f})$  is the number of connected components of  $S(\tilde{f})$ .*

We have a following example which is related to Theorem 4.1.

**Example 4.3.** Let  $T^2$  be an oriented torus and  $l$  and  $m$  longitude and meridian of  $T^2$ , respectively. Let  $\tilde{f}$  and  $\tilde{g} : T^2 \rightarrow \mathbb{R}^2$  be signed stable maps which satisfy the following properties. They do not have cusp points,  $\tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g}))$ , both signs are the same,  $\tilde{f}|l = \tilde{g}|l$  and  $\tilde{g}|m$  are plane curves whose rotation numbers equal 2 (or  $-2$ ),  $\tilde{f}|m$  is a simple closed plane curve. See Figure 3. By the theorem of Pinkall [6], immersion lifts  $f$  and  $g : T^2 \rightarrow \mathbb{R}^3$  over  $\tilde{f}$  and  $\tilde{g}$  respectively are not regularly homotopic.

Theorem 3.1 and Example 4.3 mean that if  $M \neq S^2$ , an apparent contour with sign does not determine a regular homotopy class. We need information of immersion  $\tilde{f}|(M \setminus S(\tilde{f}))$ .

## 5. REGULAR HOMOTOPY LIFT OVER A GENERIC HOMOTOPY

Let  $\tilde{f}$  and  $\tilde{g} : M \rightarrow \mathbb{R}^2$  be stable maps and  $\tilde{F} : M \times [0, 1] \rightarrow \mathbb{R}^2$  a homotopy between  $\tilde{f}$  and  $\tilde{g}$ . If  $\tilde{F}$  satisfies the following conditions, we call  $\tilde{F}$  a generic homotopy between  $\tilde{f}$  and  $\tilde{g}$  (see [5]).

- (1) There is a finite set of parameter values  $0 < t_1 < \dots < t_n < 1$  (possibly empty) in  $(0, 1)$ .
- (2) For any  $t \in (0, 1) \setminus \{t_1, \dots, t_n\}$ ,  $\tilde{F}|M \times \{t\} : M \times \{t\} \rightarrow \mathbb{R}^2$  is a stable map.
- (3) For each  $t_i$  and a sufficiently small positive value  $\varepsilon$ , the moves of apparent contours of  $\tilde{F}|M \times \{t\}$  ( $t \in (t_i - \varepsilon, t_i + \varepsilon)$ ) are classified into lips (type  $L$ ), beaks (type  $B$ ), swallowtail (type  $S$ ), cusp-fold (type  $C$ ), self-tangency (type  $K$ ) or triple point (type  $T$ ).

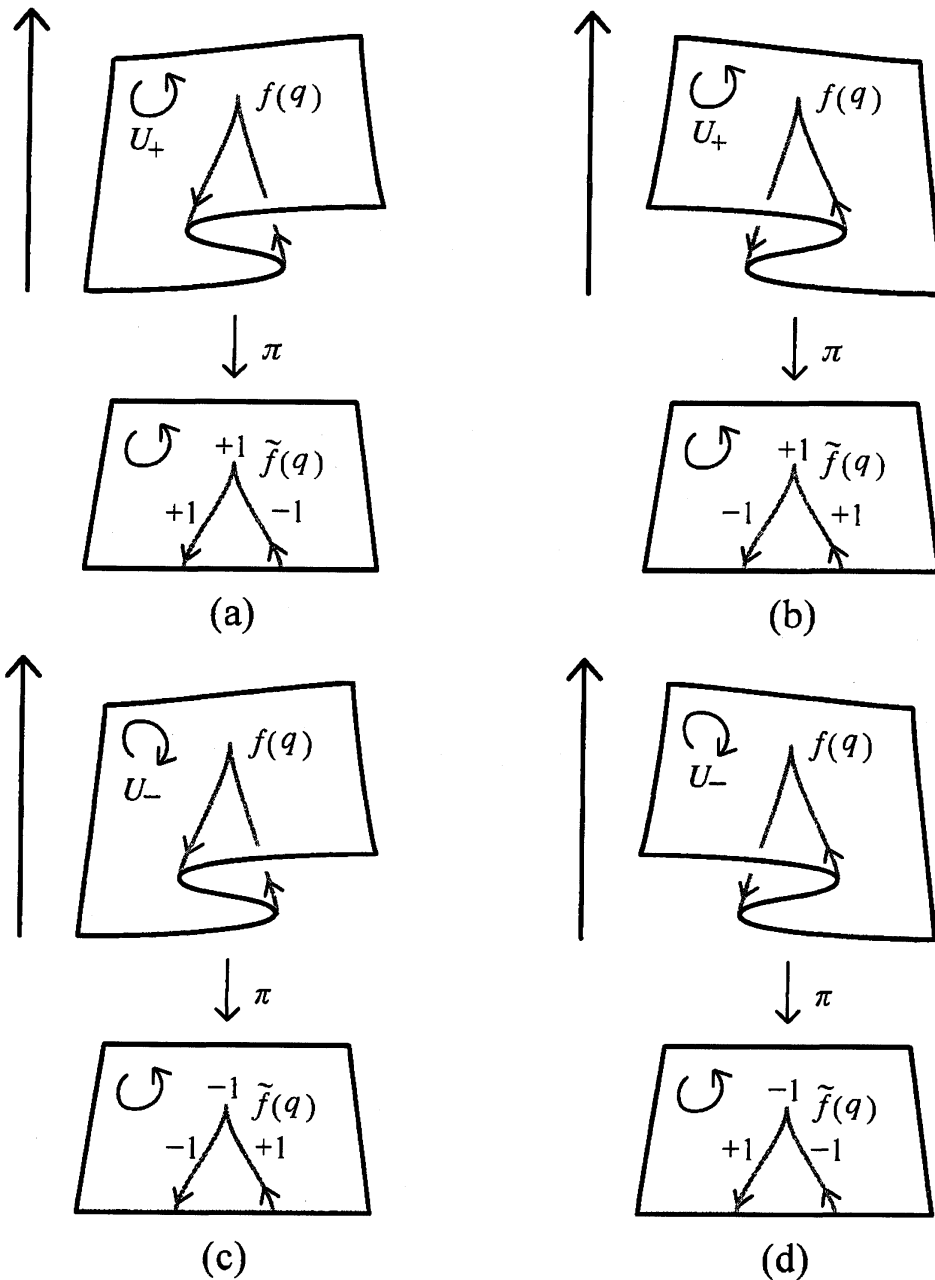


FIGURE 2. Immersion lifts if (a)  $q$  is a positive cusp and negative fold comes in, (b)  $q$  is a positive cusp and positive fold comes in, (c)  $q$  is a negative cusp and negative fold comes in, (d)  $q$  is a negative cusp and positive fold comes in.

We call each  $t_i$  a bifurcation point on a generic homotopy  $\tilde{F}$ .

Let  $\tilde{F} : M \times [0, 1] \rightarrow \mathbb{R}^2$  be a generic homotopy between signed stable maps  $\tilde{f}$  and  $\tilde{g}$  and let  $f$  and  $g$  immersion lifts over  $\tilde{f}$  and  $\tilde{g}$ , respectively. If there exists a regular homotopy  $F : M \times [0, 1] \rightarrow \mathbb{R}^3$  between  $f$  and  $g$  such that  $\pi \circ F = \tilde{F}$ , we call  $F$  a regular homotopy lift over  $\tilde{F}$ .

**Theorem 5.1.** *Let  $\tilde{f}$  and  $\tilde{g} : M \rightarrow \mathbb{R}^2$  be signed stable maps. If there exists a generic homotopy  $\tilde{F} : M \times [0, 1] \rightarrow \mathbb{R}^2$  between  $\tilde{f}$  and  $\tilde{g}$  which preserves sign*

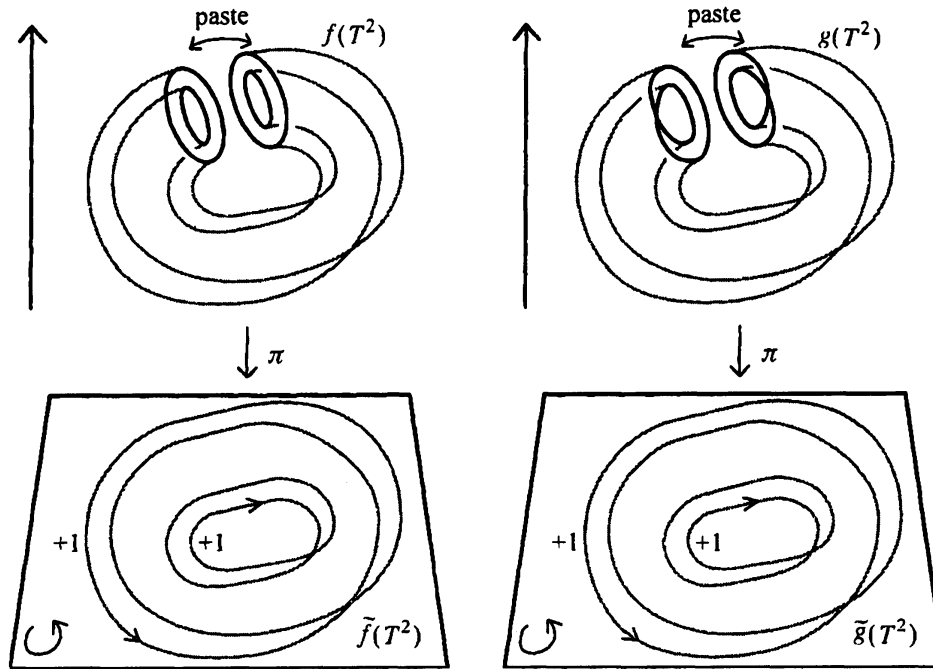


FIGURE 3. Two stable maps  $\tilde{f}$  and  $\tilde{g} : T^2 \rightarrow \mathbb{R}^2$  which satisfy that  $\tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g}))$  and both apparent contours have positive signs. But their immersion lifts  $f$  and  $g : T^2 \rightarrow \mathbb{R}^3$  are not regularly homotopic.

convention as depicted in Figures 4 and 5, then  $\tilde{F}$  has a regular homotopy lift  $F : M \times [0, 1] \rightarrow \mathbb{R}^3$ .

As an application of Theorem 5.1, we have the following example.

**Example 5.2.** If  $\tilde{f}$  and  $\tilde{g} : S^2 \rightarrow \mathbb{R}^2$  are signed stable maps such that  $\tilde{f}(S^2) = \tilde{g}(S^2) = D^2$ ,  $\tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g}))$  is a simple closed curve and the sign of  $S(\tilde{f})$  (resp.  $S(\tilde{g})$ ) is  $+1$  (resp.  $-1$ ). Then there is a generic homotopy  $\tilde{F} : S^2 \times [0, 1] \rightarrow \mathbb{R}^2$  between  $\tilde{f}$  and  $\tilde{g}$  which has a regular homotopy lift  $F : S^2 \times [0, 1] \rightarrow \mathbb{R}^3$ . See Figure 6. By the definitions of  $\tilde{f}$ ,  $\tilde{g}$ , the regular homotopy lift  $F$  over  $\tilde{F}$  corresponds to an eversion of the embedded sphere.

Our eversion in Example 5.2 is almost same as the eversion given by Francis [1]. But in his picture, self intersections of immersed spheres were not drawn. Professor Mikami Hirasawa and the author draw a regular homotopy over the generic homotopy of Figure 6, precisely. So, we can follow how self intersections move during our sphere eversion. Our eversion will appear in their preparing paper.

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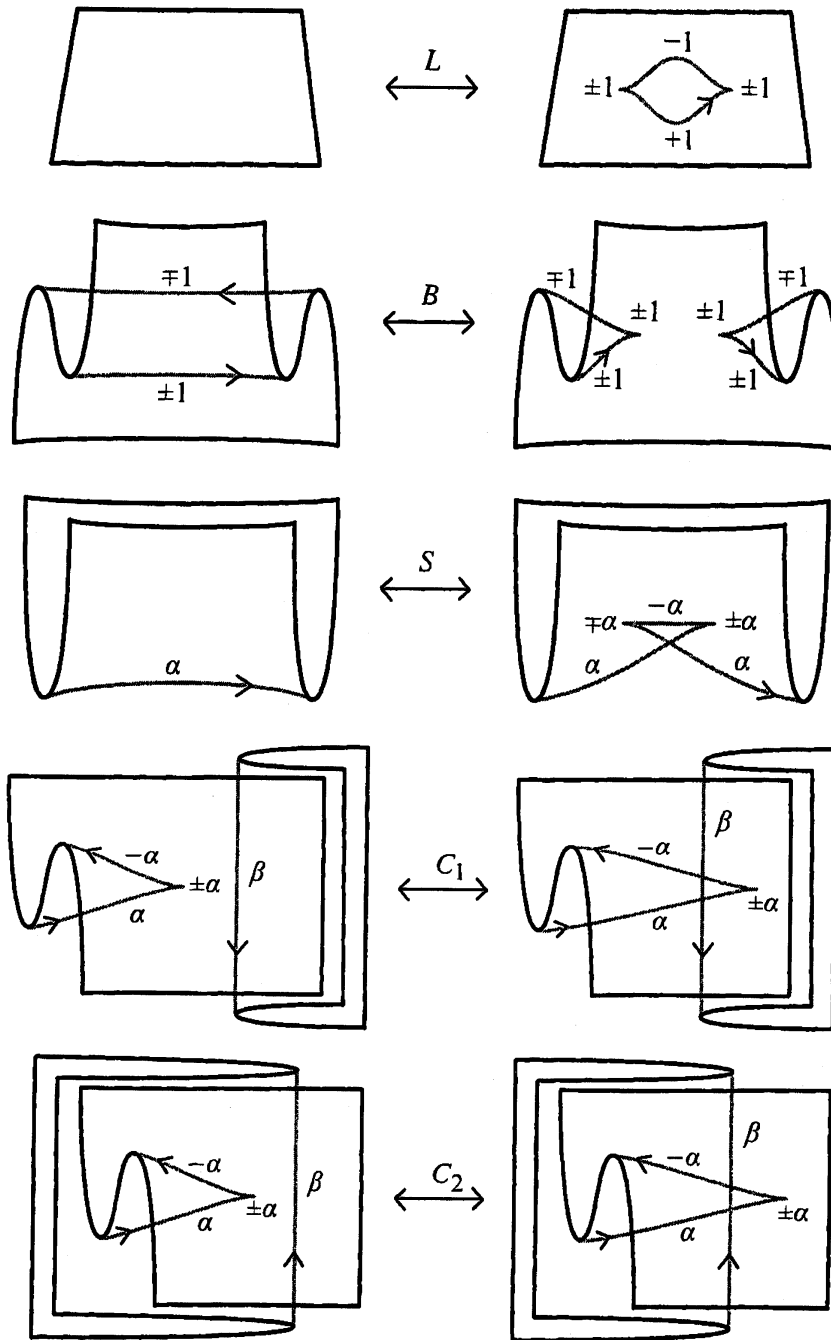


FIGURE 4. Bifurcations of type  $L$ ,  $B$ ,  $S$  and  $C$  which have regular homotopy lifts. Here,  $\alpha = \pm 1$  and  $\beta = \pm 1$  and  $\alpha$  and  $\beta$  vary independently.

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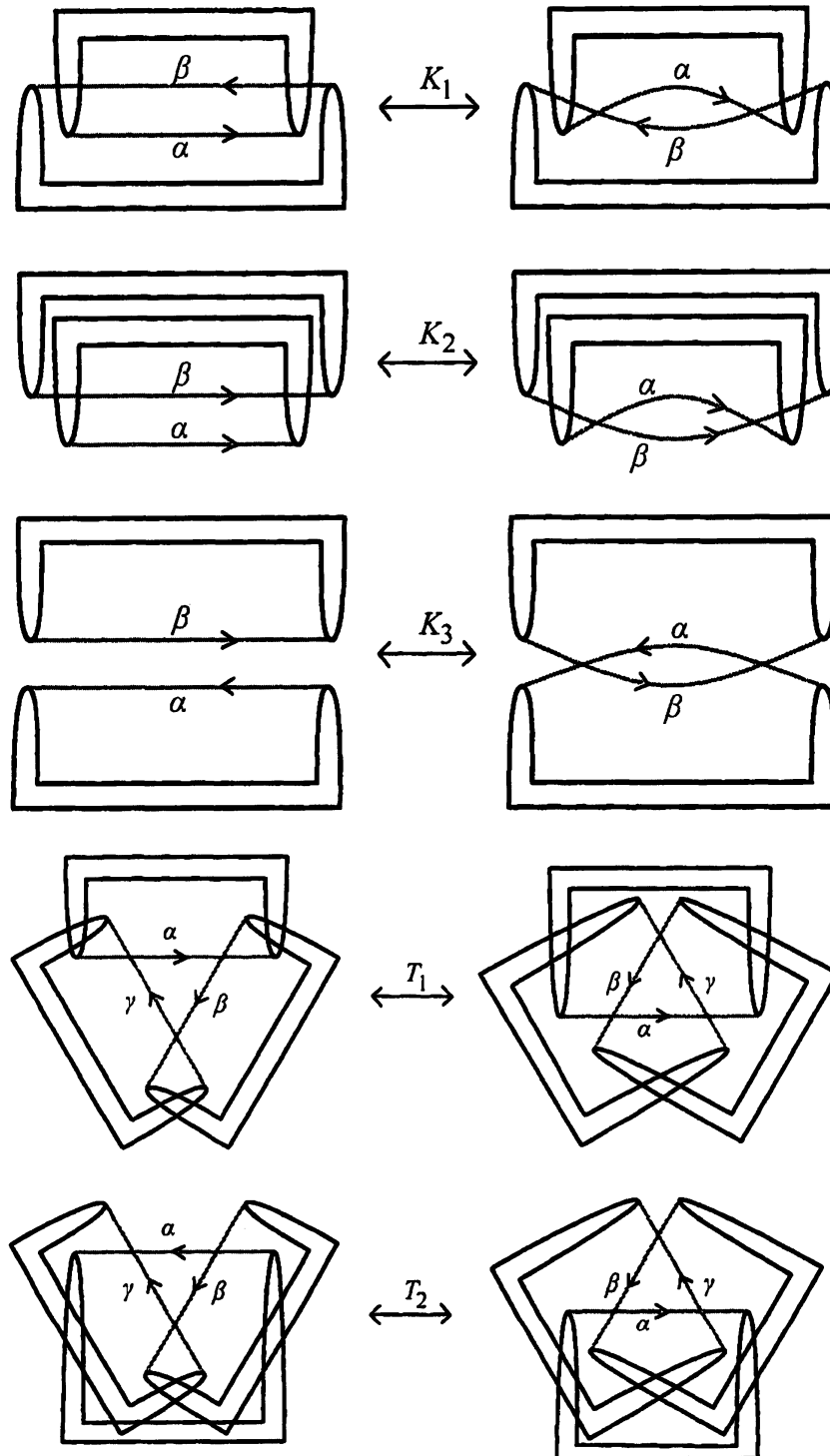


FIGURE 5. Bifurcations of type  $K$  and  $T$  which have regular homotopy lifts. Here,  $\alpha = \pm 1, \beta = \pm 1$  and  $\gamma = \pm 1$  and  $\alpha, \beta$  and  $\gamma$  vary independently.

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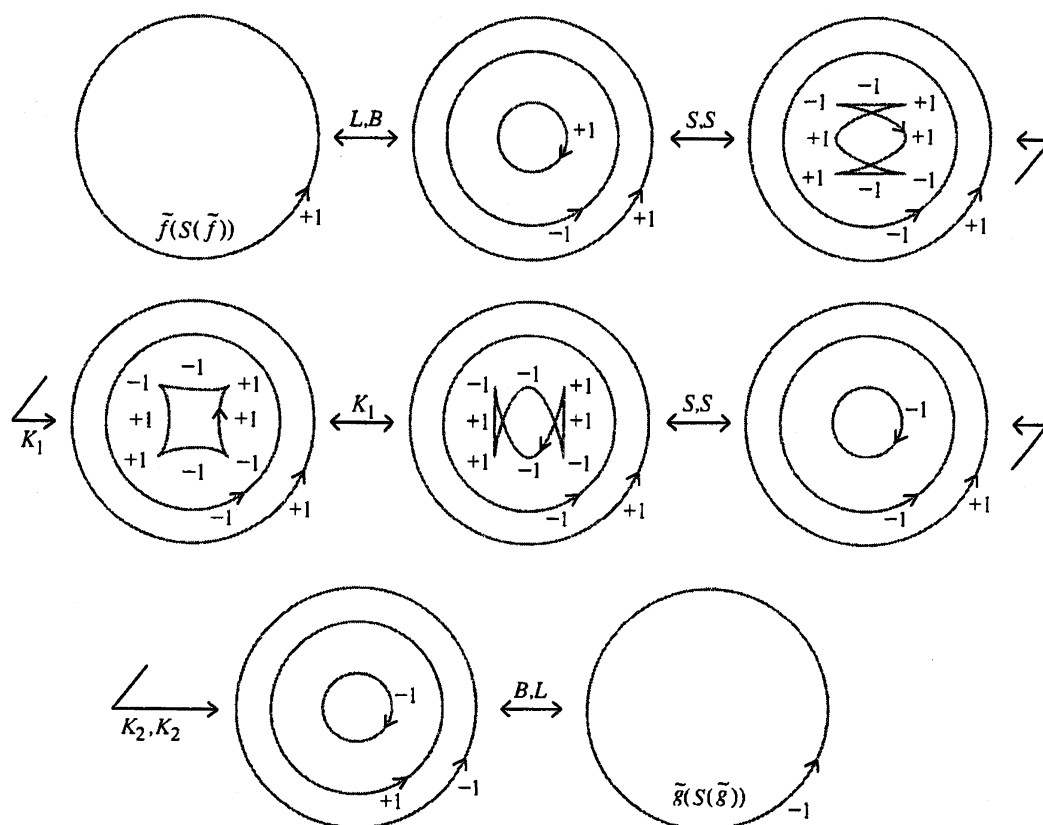


FIGURE 6. A sequence of apparent contours of a generic homotopy between  $\tilde{f}$  and  $\tilde{g} : S^2 \rightarrow \mathbb{R}^2$  which has a regular homotopy lift. This regular homotopy corresponds to a sphere eversion.